UNIT # 01

HYPERBOLIC FUNCTION

$\sin ix = i \sinh x$, $\cos ix = \cosh x$.

$\sinh(-x) = -\sinh x$, $\cosh(-x) = \cosh x$.

Formula like $\cosh^2 x - \sinh^2 x = 1$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

DERIVATIVE OF HYPERBOLIC FUNCTIONS

(1) $y = \sinh x$

$= \frac{e^x - e^{-x}}{2} \Rightarrow \frac{dy}{dx} = \frac{e^x + e^{-x}}{2} = \cosh x$

$d(\sinh x) = \cosh x$

(2) $y = \cosh x$ $\quad d(\cosh x) = \sinh x$

(3) $y = \tanh x$

$= \frac{\sinh x}{\cosh x} \Rightarrow \frac{dy}{dx} = \frac{\cosh x\, d(\sinh x) - \sinh x\, d(\cosh x)}{\cosh^2 x}$

$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x}$

$= \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

$d(\tanh x) = \operatorname{sech}^2 x$

(4) $y = \coth x$, $d(\coth x) = -\operatorname{cosech}^2 x$

(5) $y = \operatorname{sech} x$, $d(\operatorname{sech} x) = -\tanh x \operatorname{sech} x$

$= \frac{1}{\cosh x}$

$\Rightarrow \frac{dy}{dx} = \frac{\cosh x\, d(1) - 1\, d(\cosh x)}{\cosh^2 x}$

$= -\frac{\sinh x}{\cosh^2 x} = -\frac{\sinh x}{\cosh x} \cdot \frac{1}{\cosh x} = -\tanh x \operatorname{sech} x.$

(6) $y = \operatorname{cosech} x$, $d(\operatorname{cosech} x) = -\coth x \operatorname{cosech} x.$

DERIVATIVES OF INVERSE HYPERBOLIC FUNCTIONS

(1) $y = \sinh^{-1}x \Rightarrow x = \sinh y$

$$\frac{dx}{dy} = \cosh y \Rightarrow \frac{dy}{dx} = \frac{1}{\cosh y} = \pm\frac{1}{\sqrt{1+\sinh^2 y}} = \pm\frac{1}{\sqrt{1+x^2}}$$

Sign of $\frac{dy}{dx}$ is the same as that of $\cosh y$

$\cosh y$ is always +ve

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}}$$

(2) $y = \cosh^{-1}x \Rightarrow x = \cosh y$

$$\frac{dx}{dy} = \sinh y \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y} = \pm\frac{1}{\sqrt{\cosh^2 y - 1}} = \pm\frac{1}{\sqrt{x^2-1}}$$

sign of $\frac{dy}{dx}$ is same as $\sinh y$

$\cosh^{-1}x$ is always +ve $\Rightarrow$ y is always +ve.

$\Rightarrow \sinh y$ is +ve

$$\therefore \boxed{\frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}, \quad x>1}$$

(3) $y = \tanh^{-1}x \Rightarrow x = \tanh y$

Refer NOTE ON THE BACK OF THIS PAGE

$$\frac{dx}{dy} = \operatorname{sech}^2 y \Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y} = \frac{1}{1-\tanh^2 y} = \frac{1}{1-x^2}$$

$$\boxed{\therefore \frac{dy}{dx} = \frac{1}{1-x^2}, \quad |x|<1}$$

, If $y = \coth^{-1}x$, $\frac{dy}{dx} = -\frac{1}{x^2-1}$, $|x|>1$

(4) $y = \operatorname{sech}^{-1}x \Rightarrow x = \operatorname{sech} y$

$$\frac{dx}{dy} = -\operatorname{sech} y \tanh y \Rightarrow \frac{dy}{dx} = -\frac{1}{\operatorname{sech} y \tanh y} = \pm\frac{-1}{\operatorname{sech} y\sqrt{1-\operatorname{sech}^2 y}}$$

$$= -\frac{1}{x\sqrt{1-x^2}}$$

sign of $\frac{dy}{dx}$ is same as that of $\tanh y$

Since $\operatorname{sech}^{-1}x$ is always +ve $\Rightarrow$ y is positive

$\Rightarrow \tanh y$ is +ve.

$$\boxed{\frac{dy}{dx} = -\frac{1}{x\sqrt{1-x^2}} \quad 0<x<1}$$

(5) $y = \text{cosech}^{-1} x \Rightarrow x = \text{cosech}\, y$

$$\frac{dx}{dy} = -\text{cosech}\, y \coth y \Rightarrow \frac{dy}{dx} = -\frac{1}{\text{cosech}\, y \coth y}$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{-1}{\text{cosech}\, y \sqrt{\text{cosech}^2 y + 1}}$$

$$= \pm \frac{-1}{x\sqrt{x^2+1}}$$

sign of $\frac{dy}{dx}$ is same as that of $\coth y$

Now y is +ve or -ve according as x is +ve or -ve

$$\frac{dy}{dx} = \frac{-1}{x\sqrt{x^2+1}}, \quad x > 0 \quad \text{and}$$

$$= \frac{-1}{-x\sqrt{x^2+1}}, \quad x < 0$$

$$\frac{dy}{dx} = \frac{-1}{|x|\sqrt{x^2+1}}, \quad x \neq 0.$$

INTEGRATION OF HYPERBOLIC FUNCTIONS

1. $\int \sinh x \, dx = \int \frac{e^x - e^{-x}}{2} dx = \frac{1}{2}\int (e^x - e^{-x}) dx$

$$= \frac{1}{2}\left[e^x + e^{-x}\right] + c$$

$$= \cosh x$$

$$\boxed{\therefore \int \sinh x \, dx = \cosh x + c}$$

2. $\int \cosh x \, dx = \sinh x + c$
3. $\int \text{sech}^2 x \, dx = \int \frac{1}{\cosh^2 x} dx = \int \frac{4}{(e^x + e^{-x})^2} dx$

$$= \int \frac{4e^{2x}}{(e^{2x}+1)^2} dx$$

suppose $e^{2x} + 1 = u$, then $2e^{2x} dx = du$

$$= \int \frac{2du}{u^2} = -\frac{2}{u} + c = -\frac{2}{e^{2x}+1} + c = -\left[\frac{1+1+e^{2x}-e^{2x}}{e^{2x}+1}\right] + c$$

4. $\int \operatorname{cosech}^2 x \, dx = \int \frac{4}{(e^x - e^{-x})^2} dx = \int \frac{4e^{2x}}{(e^{2x}-1)^2} dx$

Suppose $e^{2x} - 1 = u$

$2e^{2x} dx = du$

$\therefore \int \operatorname{cosech}^2 x \, dx = \int \frac{2du}{u^2} = -\frac{2}{u} + c$

$= -\frac{2}{e^{2x}-1} + c = -\left[\frac{1+1+e^{2x}-e^{2x}}{e^{2x}-1}\right] + c$

$= -\left[\frac{e^{2x}+1}{e^{2x}-1} - \frac{e^{2x}-1}{e^{2x}-1}\right] + c$

$= -\frac{e^{2x}+1}{e^{2x}-1} - 1 + c$

$= -\frac{e^x + e^{-x}}{e^x - e^{-x}} + c$

$= -\coth x + c$

$\boxed{\therefore \int \operatorname{cosech}^2 x \, dx = -\coth x + c}$

5. $\int \operatorname{sech} x \tanh x \, dx = \int \frac{1}{\cosh x} \times \frac{\sinh x}{\cosh x} dx$

$= \int \frac{\sinh x}{\cosh^2 x} dx$

Suppose $\cosh x = u \Rightarrow \sinh x \, dx = du$

$\therefore \int \operatorname{sech} x \tanh x \, dx = \int \frac{du}{u^2} = -\frac{1}{u} + c$

$= -\frac{1}{\cosh x} + c$

$\boxed{\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c}$

6. $\int \operatorname{cosech} x \coth x \, dx = \int \frac{1}{\sinh x} \times \frac{\cosh x}{\sinh x} dx = \int \frac{\cosh x}{\sinh^2 x} dx$

Suppose $\sinh x = u$, $\cosh x \, dx = du$

$\therefore \int \operatorname{cosech} x \coth x \, dx = \int \frac{du}{u^2} = -\frac{1}{u} + C = -\operatorname{cosech} x + C.$

$\boxed{\therefore \int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c}$

7. $\int \frac{dx}{\sqrt{x^2+a^2}}$, put $x = a\sinh\theta \Rightarrow dx = a\cosh\theta\, d\theta$

$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2\sinh^2\theta + a^2}} = \int \frac{a\cosh\theta}{a\cosh\theta} d\theta = \int d\theta = \theta + C.$

$\therefore \boxed{\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a} + c}$

8. $\int \frac{dx}{\sqrt{x^2-a^2}}$, put $x = a\cosh\theta \Rightarrow dx = a\sinh\theta\, d\theta$

$\therefore \int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}} = \int \frac{a\sinh\theta}{a\sinh\theta} d\theta = \int d\theta.$

$= \theta + C$

$= \cosh^{-1}\left(\frac{x}{a}\right) + c$

$\therefore \boxed{\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + c}$

9. $\int \frac{dx}{a^2-x^2}$, put $x = a\tanh\theta \Rightarrow dx = a\,\text{sech}^2\theta\, d\theta$

$\therefore \int \frac{dx}{a^2-x^2} = \int \frac{a\,\text{sech}^2\theta\, d\theta}{a^2 - a^2\tanh^2\theta} = \frac{1}{a}\int \frac{\text{sech}^2\theta}{1-\tanh^2\theta} d\theta = \frac{1}{a}\int d\theta.$

$= \frac{1}{a}\theta + C = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c$

$\boxed{\therefore \int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + c}$

10. $\int \frac{dx}{x^2-a^2}$, put $x = a\coth\theta \Rightarrow dx = -a\,\text{cosech}^2\theta\, d\theta$

$\int \frac{dx}{x^2-a^2} = \int \frac{-a\,\text{cosech}^2\theta\, d\theta}{a^2\coth^2\theta - a^2} = -\frac{1}{a}\int \frac{\text{cosech}^2\theta\, d\theta}{\coth^2\theta - 1} = -\frac{1}{a}\int d\theta$

$\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\theta + C$

$\boxed{\int \frac{dx}{x^2-a^2} = -\frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + c}$

11. $\int \tanh x \, dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

put $e^x + e^{-x} = u \Rightarrow (e^x - e^{-x}) dx = du$

$\int \tanh x \, dx = \int \frac{du}{u} = \log|u| + c$

$= \log\left[e^x + e^{-x}\right] + c = \log\left[2 \frac{e^x + e^{-x}}{2}\right] + c$

$= \log\left[2\cosh x\right] + c$

$= \log|2| + \log|\cosh x| + c$

$= \log|\cosh x| + c.$

Exp. ① $\int x \,\mathrm{sech}^2 x \, dx = x\int \mathrm{sech}^2 x \, dx - \int\left[d(x)\int \mathrm{sech}^2 x \, dx\right] dx$

$= x \tanh x - \int \tanh x \, dx$

$= x \tanh x - \int \frac{\sinh x}{\cosh x} dx$

$= x \tanh x - \int \frac{dt}{t} \quad \left[\begin{array}{l}\cosh x = t \\ \sinh x \, dx = dt\end{array}\right]$

$= x \tanh x - \log|t| + c$

$= x \tanh x - \log|\cosh x| + c.$ Ans

② $\int \frac{dx}{4x^2 - 9} = \frac{1}{4}\int \frac{dx}{x^2 - \left(\frac{3}{2}\right)^2} = -\frac{1}{4}\frac{2}{3}\coth^{-1}\left(\frac{x}{3/2}\right) + c$

$= -\frac{1}{6}\coth^{-1}\left(\frac{2x}{3}\right) + c$ Ans

③ $\int \mathrm{sech}\, x \, dx = \int \frac{2 dx}{e^x + e^{-x}} = 2\int \frac{e^x dx}{e^{2x} + 1} = 2\int \frac{dt}{t^2 + 1} \left[\begin{array}{l}e^x = t \\ e^x dx = dt\end{array}\right]$

$= 2\tan^{-1} t + c$

$= 2\tan^{-1}(e^x) + c$

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$ or $-\cos^{-1} x$

$\int \frac{dx}{1+x^2} = \tan^{-1} x$ or $-\cot^{-1} x$

$\int \frac{dx}{x\sqrt{1-x^2}} = \sec^{-1} x$ or $-\mathrm{cosec}^{-1} x$.

$\int \sec^2 x \, dx = \tan x$

$\int \mathrm{cosec}^2 x \, dx = -\cot x$

$\int \sec x \tan x \, dx = \sec x$

$\int \mathrm{cosec}\, x \cot x \, dx = -\mathrm{cosec}\, x$

HIGHER ORDER DERIVATIVE

(1) $y = (ax+b)^m$, $\quad y_n = \dfrac{m!}{(m-n)!} a^n (ax+b)^{m-n}$

(2) $y = (ax+b)^{-1}$, $\quad y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

(3) $y = \log(ax+b)$, $\quad y_n = \dfrac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

(4) $y = a^{mx}$, $\quad y_n = m^n (\log a)^n a^{mx}$

(5) $y = e^{mx}$, $\quad y_n = m^n e^{mx}$

(6) $y = \cos(ax+b)$, $\quad y_n = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

(7) $y = \sin(ax+b)$, $\quad y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

(8) $y = e^{ax}\cos(bx+c)$, $\quad y_n = r^n e^{ax}\cos(bx+c+n\phi)$

where $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$

(9) $y = e^{ax}\sin(bx+c)$, $\quad y_n = r^n e^{ax}\sin(bx+c+n\phi)$

where $r = \sqrt{a^2+b^2}$, $\phi = \tan^{-1}\left(\frac{b}{a}\right)$.

(10) Find y_n for $y = \cos mx - \sin mx$

Ans. $y_n = m^n\left[1-(-1)^n \sin 2mx\right]^{1/2}$

(11) Find y_n for $y = e^{2x}\cos x \sin^2 2x$.

Ans $y_n = \dfrac{5^{n/2}}{2} e^{2x}\cos\left(x+n\tan^{-1}\frac{1}{2}\right) - \dfrac{29^{n/2}}{2} e^{2x}\cos\left(5x+n\tan^{-1}\frac{5}{2}\right)$

$\qquad - \dfrac{13^{n/2}}{4} e^{2x}\cos\left(3x+n\tan^{-1}\frac{3}{2}\right)$

Q-10 $y = \cos mx - \sin mx$

Sol. $y_n = m^n\left[\cos\left(mx + \frac{n\pi}{2}\right) - \sin\left(mx + \frac{n\pi}{2}\right)\right]$

$= m^n\left[\left[\cos\left(mx + \frac{n\pi}{2}\right) - \sin\left(mx + \frac{n\pi}{2}\right)\right]^2\right]^{1/2}$

$= m^n\left[1 - 2\sin\left(mx + \frac{n\pi}{2}\right)\cos\left(mx + \frac{n\pi}{2}\right)\right]^{1/2}$

$= m^n\left[1 - \sin(2mx + n\pi)\right]^{1/2}$

$= m^n\left[1 - (-1)^n \sin 2mx\right]^{1/2}$ Ans

Q-11 $y = e^{2x}\cos x \sin^2 2x$

Sol. $y = e^{2x}\cos x\left(1 - \frac{\cos 4x}{2}\right)$

$= \frac{1}{2}e^{2x}(\cos x - \cos x \cos 4x)$

$= \frac{1}{2}e^{2x}\left[\cos x - \frac{1}{2}(\cos 5x + \cos 3x)\right]$

$= \frac{1}{2}e^{2x}\cos x - \frac{1}{4}e^{2x}\cos 5x - \frac{1}{4}e^{2x}\cos 3x.$

Leibnitz's Theorem

Suppose u and v are nth derivable functions then

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{n-2} v_2 + \cdots + u v_n$$

Exp. $x = \cos\left(\frac{1}{m}\log y\right)$, find $y_n(0)$

Sol. $\Rightarrow \cos^{-1}x = \frac{\log y}{m}$

$\Rightarrow \log y = m\cos^{-1}x$

$\Rightarrow y = e^{m\cos^{-1}x}$

$y_1 = e^{m\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}}\right)\times m$

$\Rightarrow (1-x^2)y_1 = m^2 y^2$

Differentiating again $(1-x^2)y_2 - xy_1 = m^2 y$

Diff n times $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

$y(0) = e^{\frac{m\pi}{2}}$, $y_1(0) = -me^{\frac{m\pi}{2}}$, $y_2(0) = m^2 e^{m\pi/2}$

$y_{n+2}(0) = (n^2+m^2)\, y_n(0)$.

$y_3(0) = -(1^2+m^2)\, m e^{m\pi/2}$

$y_4(0) = e^{m\pi/2} m^2 (m^2+2^2)$

$y_5(0) = -e^{\frac{m\pi}{2}} m(m^2+3^2)$

$\vdots$

$$y_n(0) = \begin{cases} e^{\frac{m\pi}{2}} m^2(m^2+2^2)\cdots(m^2+(n-2)^2) & \text{if } n \text{ is even} \\ -e^{\frac{m\pi}{2}} m(m^2+1^2)\cdots(m^2+(n-2)^2), & \text{If } n \text{ is odd} \end{cases}$$

Exp. ① $y = x\log(x-1)$,

$$y_n = \frac{(-1)^{n-2}(n-2)!}{(x-1)^n}\cdot(x-n)$$

② ~~pt cosl~~ $y = e^{m\sin^{-1}x}$, find $y_n(0)$

$$y_1 = e^{m\sin^{-1}x} \times \frac{1\times m}{\sqrt{1-x^2}}$$

Sol. $(1-x^2)y_1^2 = m^2y^2$ — ①

$(1-x^2)y_2 - xy_1 = m^2y$ — ②

Diff. n times

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$

put $x=0$

$y(0)=1,\ y_1(0)=m,\ y_2(0)=m^2,\ y_{n+2}(0)=(n^2+m^2)y_n(0)$

$n = 1, 2, 3, 4 \ldots$

$$y_n(0) = \begin{cases} m^2(2^2+m^2)(4^2+m^2)\ldots[(n-2)^2+m^2], & n \text{ is even} \\ m(1^2+m^2)(3^2+m^2)\ldots[(n-2)^2+m^2], & n \text{ is odd} \end{cases}$$

③ $y = \cos(m\sin^{-1}x)$ PT

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Sol. $y_1 = -\sin(m\sin^{-1}x)\dfrac{m}{\sqrt{1-x^2}}$

$\Rightarrow (1-x^2)y_1^2 = m^2(1-y^2)$

Diff. again $(1-x^2)y_2 - xy_1 + m^2y = 0$

Diff. n times

we get answer.

INDETERMINATE FORMS

If $\lim_{x\to a} F(x) = 0$, $\lim_{x\to a} f(x) = 0$, then

$$\lim_{x\to a} \frac{F(x)}{f(x)} = \frac{0}{0}$$ is called indeterminate form

then $$\lim_{x\to a} \frac{F'(x)}{f'(x)} = l \Rightarrow \lim_{x\to a} \frac{F(x)}{f(x)} = l.$$

Other indeterminate forms are

$$\frac{\infty}{\infty},\ \infty - \infty,\ 0^0,\ \infty^0,\ 1^\infty,\ 0.\infty$$

(1) $\frac{\infty}{\infty}$, form if $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} F(x)$, then

$$\lim_{x\to a} \frac{F'(x)}{f'(x)} = l \Rightarrow \lim_{x\to a} \frac{F(x)}{f(x)} = l$$

(2) $\infty - \infty$, form.

Consider $\lim_{x\to a} [f(x) - F(x)]$, when

$$\lim_{x\to a} f(x) = \infty = \lim_{x\to a} F(x)$$

we write $$f(x) - F(x) = \left[\frac{1}{F(x)} - \frac{1}{f(x)}\right] \div \frac{1}{f(x)F(x)}$$

So that the numerator $\left[\frac{1}{F(x)} - \frac{1}{f(x)}\right]$ is zero

and the denominator $\frac{1}{f(x)F(x)}$ is zero.

(3) $0^0, 1^\infty, \infty^0$

Consider $\lim_{x\to a}\left[f(x)^{F(x)}\right]$

Different cases are

$\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} F(x) = 0$

$\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} F(x) = \infty$

$\lim_{x\to a} f(x) = \infty$, $\lim_{x\to a} F(x) = 0$

For all above cases, we write

$$y = f(x)^{F(x)}$$

Taking log both sides, we get

$$\log y = \log f(x)^{F(x)}$$

$$= F(x) \log f(x)$$

$$\lim_{x\to a} \log y = \lim_{x\to a} F(x) \log f(x)$$

$$\log \lim_{x\to a} y = \lim_{x\to a} F(x) \log f(x)$$

In all above cases RHS Take indeterminate form $0.\infty$

Suppose $\lim_{x\to a} F(x) \log f(x) = l$

$\therefore \log \lim_{x\to a} y = l \Rightarrow \boxed{\lim_{x\to a} y = e^l}$

(4) $0.\infty$

Consider $\lim_{x\to a} f(x).F(x)$, where $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} F(x) = \infty$

we write $f(x).F(x) = \dfrac{F(x)}{1/f(x)}$ OR $\dfrac{f(x)}{1/F(x)}$, which assumes $\frac{0}{0}$ or $\frac{\infty}{\infty}$ form.

Exp. Ex. (1) Evaluate $\lim\limits_{x\to 2} \dfrac{\sin(x^2-4)}{x-2}$

Solution: $\lim\limits_{x\to 2} \dfrac{\sin(x^2-4)}{x-2} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 2} \frac{\cos(x^2-4).2x}{1} = 4 \text{ Ans}$$

(2) Evaluate $\lim\limits_{x\to 0} \dfrac{e^x + \log(1-x) - 1}{\tan x - x}$

Solution: $\lim\limits_{x\to 0} \dfrac{e^x + \log(1-x) - 1}{\tan x - x} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{e^x - \frac{1}{1-x}}{\sec^2 x - 1} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x(1-x) - 1}{(1-x)\tan^2 x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x(1-x) - 1}{(1-x)x^2} \times \lim_{x\to 0} \frac{x^2}{\tan^2 x}$$

$$= \lim_{x\to 0} \frac{e^x(1-x) - 1}{x^2 - x^3} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{e^x(1-x) - e^x}{2x - 3x^2}$$

$$= \lim_{x\to 0} \frac{-xe^x}{2x - 3x^2} = \lim_{x\to 0} \frac{-e^x}{2-3x} = -\frac{1}{2} \text{ Ans}$$

(3) Find a, b, c, so that $\lim\limits_{x\to 0} \dfrac{ae^x - 2b\cos x + 3ce^{-x}}{x\sin x} = 2$

Solution. $\lim\limits_{x\to 0} \dfrac{ae^x - 2b\cos x + 3ce^{-x}}{x\sin x}$

Since the denominator is 0 at $x = 0$

$\therefore \quad a - 2b + 3c = 0 \quad \text{——} (1)$

with this condition our limit reduces to $\frac{0}{0}$ form.

$$= \lim_{x\to 0} \frac{ae^{x} - 2b\cos x + 3ce^{-x}}{x \cdot x} \times \lim_{x\to 0} \frac{x}{\sin x}$$

$$= \lim_{x\to 0} \frac{ae^{x} - 2b\cos x + 3ce^{-x}}{x^2}$$

$$= \lim_{x\to 0} \frac{ae^{x} + 2b\sin x - 3ce^{-x}}{2x}$$

$$\Rightarrow a - 3c = 0 \quad \text{———} (2)$$

with this condition our limit reduces to $\frac{0}{0}$ form.

So, using L'Hospital rule, we get

$$= \lim_{x\to 0} \frac{ae^{x} + 2b\cos x + 3ce^{-x}}{2}$$

$$\Rightarrow \frac{a+2b+3c}{2} = 2$$

$$\Rightarrow a+2b+3c = 4 \quad \text{———} (3)$$

Solving equations (1), (2) and (3), we get

$a = 1, c = \frac{1}{3}, b = 1.$ Ans

(4) Evaluate $\lim_{x\to 0} \frac{\log(\log(1-3x^2))}{\log(\log(\cos 2x))}$

Solution: $\lim_{x\to 0} \frac{\log(\log(1-3x^2))}{\log(\log(\cos 2x))} \quad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0} \frac{\frac{1}{\log(1-3x^2)}\left(\frac{-6x}{1-3x^2}\right)}{\frac{1}{\log(\cos 2x)}\left(-\frac{2\sin 2x}{\cos 2x}\right)}$$

$$= \frac{3}{2} \lim_{x\to 0} \frac{\log(\cos 2x)}{\log(1-3x^2)} \quad \left(\frac{0}{0}\right)$$

$$= \frac{3}{2} \lim_{x\to 0} \frac{\frac{1}{\cos 2x}(-2\sin 2x)}{\frac{-6x}{1-3x^2}} = 1 \quad \text{Ans}$$

(5) Evaluate $\lim_{x\to 0}\left(\frac{1}{2x^2}-\frac{\cot^2 x}{2}\right)$

Solution $\lim_{x\to 0}\left(\frac{1}{2x^2}-\frac{\cot^2 x}{2}\right)$ $(\infty-\infty)$

$$= \lim_{x\to 0}\left(\frac{1}{2x^2}-\frac{1}{2\tan^2 x}\right)$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^2\tan^2 x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\tan^2 x - x^2}{2x^4} \quad \left(\frac{0}{0}\right)$$

using L Hospital's rule, we get $= \frac{1}{3}$ Ans.

(6) Evaluate $\lim_{x\to 1} (4-4x^2)^{\frac{1}{\log(2-2x)}}$

Solution $\lim_{x\to 1} (4-4x^2)^{\frac{1}{\log(2-2x)}}$ (0^0)

Suppose $y = (4-4x^2)^{\frac{1}{\log(2-2x)}}$

Taking log both sides

$$\log y = \log (4-4x^2)^{\frac{1}{\log(2-2x)}}$$

$$\log y = \frac{1}{\log(2-2x)} \log(4-4x^2)$$

$$\lim_{x\to 1} \log y = \lim_{x\to 1} \frac{\log(4-4x^2)}{\log(2-2x)}$$

$$\log \lim_{x\to 1} y = \lim_{x\to 1} \frac{\log(4-4x^2)}{\log(2-2x)} \quad \left(\frac{\infty}{\infty}\right)$$

Using L Hospital's rule, we get

$$\log \lim_{x\to 1} y = 1$$

$$\Rightarrow \lim_{x\to 1} y = e^1 = e \quad \text{Ans}$$

(7) Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{5}{3x^2}}$

Solution:

$$\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{5}{3x^2}} \quad 1^{\infty}$$

Suppose $y = \left(\frac{\tan x}{x}\right)^{\frac{5}{3x^2}}$

Taking log both sides, we get

$$\log y = \log\left(\frac{\tan x}{x}\right)^{5/3x^2}$$

$$\log y = \frac{5}{3x^2}\log\left(\frac{\tan x}{x}\right)$$

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\frac{5\log\left(\frac{\tan x}{x}\right)}{3x^2}$$

$$\log \lim_{x\to 0} y = \lim_{x\to 0}\frac{5\log\left(\frac{\tan x}{x}\right)}{3x^2} \quad \left(\frac{0}{0}\right)$$

Using L Hospital rule, we get

$$\log \lim_{x\to 0} y = 5/9$$

$$\Rightarrow y = e^{5/9} \quad \text{Ans.}$$

(8) Evaluate $\lim_{x\to 0}(\cot x)^{\sin 2x}$

Solution: $\lim_{x\to 0}(\cot x)^{\sin 2x} \quad (\infty^0)$

Suppose $y = (\cot x)^{\sin 2x}$

Taking log both sides, we get

$$\log y = \log(\cot x)^{\sin 2x}$$

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\log(\cot x)^{\sin 2x}$$

$$\log \lim_{x\to 0} y = \lim_{x\to 0}\sin 2x \log(\cot x)$$

$$= \lim_{x\to 0}\frac{\log(\cot x)}{\operatorname{cosec} 2x} \quad \left(\frac{\infty}{\infty}\right)$$

Using L Hospital's rule, we get

$$\log \lim_{x \to 0} y = 0$$

$$\Rightarrow \lim_{x \to 0} y = e^0$$

$$\Rightarrow \lim_{x \to 0} y = 1 \quad \underline{\underline{\text{Ans}}}$$